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Instantons on quivers and orientifolds

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Abstract: We compute the prepotential for gauge theories descending from $\mathcal{N} = 4$ SYM via quiver projections and mass deformations. This accounts for gauge theories with product gauge groups and bifundamental matter. The case of massive orientifold gauge theories with gauge group SO/Sp is also described. In the case with no gravitational corrections the results are shown to be in agreement with Seiberg-Witten analysis and previous results in the literature.

Keywords: Topological Field Theories, Nonperturbative Effects, Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories.

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1. Introduction

Localization techniques have proved to be a very powerful tool in the analysis of supersymmetric gauge theories (SYM). For some years there was a very slow progress in multi-instanton computations due to the difficulty of properly treating the ADHM constraints (see [1] for a review and a complete list of references). Early attempts to introduce localization were not very fruitful [2, 3] due to the non-compactness of the moduli space of gauge connections.[1] At last a proper treatment of these problems was suggested [5] (see also [6, 7]). The key new ingredient is the introduction of a deformation (by turning gravitational backgrounds) of the theory which localizes the ADHM integrals around certain critical point of a suitable defined equivariant differential. The ADHM construction of self-dual connections in the presence of gravitational backgrounds have been worked it out in [8].

The computation of the partition function for a SYM theory can then be carried out by expanding around critical points and keeping only quadratic terms in the fluctuations. The final result is the inverse of the determinant of the Hessian. The computation of the eigenvalues of this determinant becomes now the real problem. The symmetries which deform the theory to study the equivariant cohomology come now to the rescue. Since the saddle points are critical points for these symmetries, the tangent space can be decomposed

[1] For a comparison with analogous problems for the moduli spaces of punctured Riemann surfaces see [4].

in terms of the eigenvalues with respect to the deforming symmetries. This allows the computation of the character and, as a consequence, of the eigenvalues of the Hessian [9]. This computation was carried out for $\mathcal{N} = 2, 2^*, 4$ theories in [6, 7]. See [10] for a treatment of the localization formula on a supermanifold.

The character is an extremely versatile function of the eigenvalues. Further symmetries of the theory can be incorporated into it with no need of further computation. A demonstration of this statement is given in this paper in which we compute the partition function of some quiver gauge theories and in a companion paper in which we compute the partition functions of $\mathcal{N} = 2, 2^*, 4$ SYM theories on ALE manifolds [11].

We have now to specify which symmetries we are referring to. The moduli spaces of gauge connections for the $\mathcal{N} = 4$ SYM naturally arise in the D(-1)-D3-brane system [12, 13]. The presence of the D3-brane naturally splits the SO(10), originally acting on the ten dimensional space, into SO(4)$\times$SO(6). SO(6) is the internal symmetry of the $\mathcal{N} = 4$ SYM. To get a theory with a smaller number of supersymmetries we can mod the compactified dimensions by a discrete group $\mathbb{Z}_p$ [16, 15]. Modding instead the space-time by the discrete group we find ALE manifolds [11]. The crucial point is that these discrete groups act trivially on the fixed points of the deformed theory. This implies that the new results can be extracted from the $\mathbb{R}^4$ computations of [6, 7] by doing proper projections on the tangent spaces.

In this paper we try to present these facts without referring to the formalism of the ADHM construction of gauge connections which is the standard point of view that can be found in literature. Hoping to make our results clearer to the string-oriented reader, our point of view will be strictly the one of the system of k D(-1) and N D3-branes. In section 2 we describe the massless content of this system and derive the BRST transformations on the moduli space from SUSY transformations in two dimensions (that we discuss in appendix A for completeness). We then discuss the computations of the integral on the moduli space and of the character. In the first part of section 3 we apply an orbifold projection $\Gamma = \mathbb{Z}_p$ on the moduli space and derive the corresponding character. This accounts for quiver theories[2] with product gauge group $\prod_{q=0}^{p-1} \mathrm{U}(N_q)$ and matter in the bifundamental representations $(N_q, \bar{N}_{q+1}), (N_{q+1}, \bar{N}_q)$. The second part deals with explicit computations. We are able to treat a wide range of models: massive or massless with various matter content. The largest formulae are confined in appendices B, C. Appendix B is a check against existing literature [17, 18, 19]. Finally in section 4, instead of modding by a discrete group, we introduce an orientifold plane in the D(-1)-D3-brane system. The resulting SYM has Sp or SO gauge group and has been recently studied in [20, 21]. The character for this system quickly leads to the results for the cases with mass in the adjoint and fundamental representations. In addition, the character formulae derived here compute directly the determinant at the fixed point as a product over $4kN$ (the dimension of the moduli space) eigenvalues in contrast with the approach followed in [20, 21] where the same determinant is expressed in terms of a residue of a ratio of $4kN + 4k^2$ and $4k^2$ eigenvalues. We believe that this simplification can be useful for future developments.

[2]See [14] for an early discussion on quivers.

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2. D-instantons

2.1 ADHM manifold

The moduli space of self-dual solutions of U(N) YM-equations in four dimensions is elegantly described by a real $4kN$-dimensional hypersurface embedded in a $4kN + 4k^2$-dimensional ambient space via the ADHM constraints. Alternatively one can think of the ADHM constraints as the D and F-flatness conditions of an auxiliary gauge theory of U(k) matrices. The relevant theory is the U(k) gauge theory describing the low energy dynamics of a stack of k D(-1)-branes superposed to a stack of N D3-branes. D3-brane descriptions of four dimensional gauge theories are by now of common use. In this formalism a U(N) gauge theory is realized by the lowest modes of the open strings ending on a stack of N D3-branes. These include an adjoint vector field and adjoint scalars whose N eigenvalues parameterize the positions of the $D3$ branes in the transverse space. The number of flat transverse directions can be chosen to be $2, 6$ according to whether one considers an $\mathcal{N} = 2, 4$ gauge theory. The $\mathcal{R}$-symmetry groups $\mathrm{SU}(2), \mathrm{SO}(6)$ act as rotations on this internal space. D(-1)-branes are to be thought of as instantons in the D3-brane gauge theory with multi-instanton moduli now associated to open strings ending on the D(-1)-instanton stack.

We start by considering the $\mathcal{N} = 4$ supersymmetric gauge theory. The $\mathcal{N} = 4$ SYM lagrangian follows from the dimensional reduction of the $\mathcal{N} = 1$ SYM in $D = 10$ down to the $D = 4$ worldvolume of the D3-branes. The presence of the D3-branes breaks the Lorentz symmetry of the ten-dimensional space seen by the D(-1)-instantons down to $\mathrm{SO}(4) \times \mathrm{SO}(6)_{\mathcal{R}}$. The resulting gauge theory living on the D(-1)-instanton worldvolume follows from the dimensional reduction down to zero dimension of a $\mathcal{N} = 1$ U(k) SYM theory in $D = 6$ with an adjoint (D(-1)-D(-1) open strings) and N fundamental (D(-1)-D3 open strings) hypermultiplets. In the notation of $\mathcal{N} = 1$ in $D = 4$ this corresponds to a U(k) vector multiplet, three adjoint chiral multiplets and $2N$ fundamental chiral multiplets.

For our purposes it is convenient to view the above multiplets from a $D = 2$ perspective. The two-dimensional plane where the $D = 2$ gauge theory lives will be parameterized by a complex scalar field ϕ. The choice of the plane is a matter of convention, it corresponds to choose a $\mathcal{N} = 2$ subalgebra inside $\mathcal{N} = 4$. The field ϕ plays the role of a connection in the resulting $D = 2$ gauge theory and it will be crucial in the discussion of localization below. Localization is based on the existence of a BRST current Q [24, 2, 15] in the moduli space. The charge Q is BRST in the sense that it squares to zero up to a symmetry transformation parameterized by ϕ. Deformations of the gauge theory like vevs or gravitational backgrounds will correspond to distribute the k D(-1)-instantons along the ϕ-plane.

Let us start first by describing the field content of the theory. For convenience we use the more familiar $D = 4$ $\mathcal{N} = 1$ notation to describe the $D = 6$ starting gauge theory. The starting point is a $\mathcal{N} = 1$ U(k) gauge theory in $D = 4$ with vector multiplet $V = (\phi, B_4|\eta, \chi_{\mathbb{R}}, \mathcal{M}_4|H_{\mathbb{R}})$, three adjoint chiral multiplets $C_{\ell=1,2,3} = (B_\ell|\mathcal{M}_\ell, \chi_{\ell 4}|H_{\ell 4})$ and $2N$ chiral multiplets in the fundamental $C_{\dot\alpha} = (w_{\dot\alpha}|\mu_{\dot\alpha}, \mu_{\dot a}|H_{\dot a})$, $\dot\alpha = 1, 2$, $\dot a = 2 + \dot\alpha$. Here all fields are complex except $\chi_{\mathbb{R}}, \eta, H_{\mathbb{R}}$. In particular the five complex scalars describing the instanton position in the ten-dimensional space are denoted by ϕ, B_ℓ.[3] ϕ and B_4 describe the four real components of the vector field, while $w_{\dot\alpha} = (I, J^\dagger)$ describe D(−1)-D3 string modes.[4] Fermions and auxiliary fields (here denoted by H) complete the supermultiplets. The BRST transformations of these fields are obtained in two steps. First we take the variations of $\mathcal{N}=1$ supersymmetry in $D=4$ and compactify to $D=2$ [26]. The resulting theory has $\mathcal{N}=(2,2)$ supersymmetry. The content of the $\mathcal{N}=(2,2)$ vector and chiral multiplets, their SUSY transformations and the translation to instanton variables are given in appendix A. Second the BRST charge, Q, is defined by choosing $\bar\epsilon_- = \epsilon_- = 0$, $\bar\epsilon_+ = -\epsilon_+ = i\xi$ in (A.1), (A.2)[5] and further compactifying to zero dimension. The covariant derivatives $-4i\nabla_{--}$ are then replaced by the connection ϕ. From (A.3) one finds

$$\begin{aligned}
Q\,B_\ell &= \mathcal{M}_\ell & Q\,\mathcal{M}_\ell &= [\phi, B_\ell]\\
Q\,w_{\dot\alpha} &= \mu_{\dot\alpha} & Q\,\mu_{\dot\alpha} &= \phi\, w_{\dot\alpha}\\
Q\,\mu_{\dot a} &= H_{\dot a} & Q\,H_{\dot a} &= \phi\,\mu_{\dot a}\\
Q\,\chi_{[\ell_1\ell_2]} &= H_{[\ell_1\ell_2]} & Q\,H_{[\ell_1\ell_2]} &= [\phi, \chi_{[\ell_1\ell_2]}]\\
Q\,\chi_{\mathbb{R}} &= H_{\mathbb{R}} & Q\,H_{\mathbb{R}} &= [\phi, \chi_{\mathbb{R}}]\\
Q\,\bar\phi &= \eta & Q\,\eta &= [\phi, \bar\phi]\\
Q\,\phi &= 0\,.
\end{aligned} \tag{2.1}$$

Here $\ell_{1,2} = 1, \ldots, 4$ and $\chi_{[\ell_1\ell_2]}$,$H_{[\ell_1\ell_2]}$ are antisymmetric two-tensors with six real components.

The H are auxiliary fields (D or F terms) implementing the generalized ADHM constraints

$$\begin{aligned}
\mathcal{E}_{\mathbb{R}} &= [B_\ell, B_\ell^\dagger] + II^\dagger - J^\dagger J - \zeta = 0\\
\mathcal{E}_{12} &= [B_1, B_2] + [B_3^\dagger, B_4^\dagger] + IJ = 0\\
\mathcal{E}_{13} &= [B_1, B_3] - [B_2^\dagger, B_4^\dagger] = 0\\
\mathcal{E}_{14} &= [B_1, B_4] + [B_2^\dagger, B_3^\dagger] = 0\\
\mathcal{E}_1 &= B_3 I - B_4^\dagger J^\dagger = 0\\
\mathcal{E}_2 &= B_4 I + B_3^\dagger J^\dagger = 0
\end{aligned} \tag{2.2}$$

with $\mathcal{E}_{AB} \equiv \frac{1}{2}\epsilon_{ABCD}\mathcal{E}^\dagger_{CD}$. It is convenient to collect all bosonic (except ϕ) and fermionic fields in the l.h.s. of (2.1) into two vectors: the (super)moduli are those that appear in the l.h.s. of (2.1), while the Q-differentials or tangent vectors are those in the r.h.s.

$$\begin{aligned}
\vec m &= (B_\ell, w_{\dot\alpha}) & \vec\Psi &= Q\,\vec m\\
\vec\chi &= (\chi_{[\ell_1\ell_2]}, \mu_{\dot a}) & \vec H &= Q\,\vec\chi
\end{aligned} \tag{2.3}$$

[3] In the $D=6$ notation the six-dimensional vector χ_a is built out of the three complex scalars $\phi, B_{3,4}$. See [1] for more details.

[4] $(I, J^\dagger)$ are $k \times N$ matrices which constitute the first N rows in the ADHM matrix. In the $\mathcal{N}=4$ theory, we are currently describing, the usual ADHM construction needs to be generalized. See [24, 15] for a discussion of this point.

[5] SUSY and BRST transformations (δ and Q respectively) are put in relation choosing $\delta = \xi Q$.

with $Q^2 A = \phi \cdot A$ where $\phi \cdot A$ is given by $[\phi, A]$, ϕA or $A\phi$ according to whether A transform in the adjoint, fundamental or antifundamental representation of $\mathrm{U}(k)$.

In terms of these variables the multi-instanton action reads [15, 24]

$$S = Q \,\mathrm{Tr}\left[\frac{1}{4}\,\eta[\phi,\bar\phi] + \frac{1}{2}\,\mathcal{E}_{\mathbb{R}}\,\chi_{\mathbb{R}} + \vec{\mathcal{E}}\,\vec\chi + \vec H\,\vec\chi + \vec m\,(\bar\phi\cdot\vec{\bar\Psi})\right]. \tag{2.4}$$

The convention for the vector product is $\vec\chi\,\vec H \equiv \sum_{i<j} \chi^\dagger_{[\ell_i\ell_j]} H_{[\ell_i\ell_j]}$.

Less supersymmetric multi-instanton actions are then found via suitable orbifold projections or mass deformations of the original $\mathcal{N}=4$ theory. Additional $\mathcal{N}=2$ fundamental matter can be added by introducing D7-branes. This leads to extra instanton moduli arising from D(-1)-D7 strings and transforming as

$$Q\,\mathcal{K}_f = H_f \qquad Q\,H_f = \phi\,\mathcal{K}_f\,. \tag{2.5}$$

2.2 D-instanton partition

The next step is to evaluate the centered D-instanton partition function

$$\mathcal{Z}_k = \frac{1}{\mathrm{vol}\,\mathrm{U}(k)} \int d\phi\, d\vec m\, d\vec H\, d\vec{\bar\Psi}\, d\vec\chi\, (d\bar\phi\;dH_{\mathbb{R}} d\chi_{\mathbb{R}}\, d\eta)\, e^{-S} = \int_M e^{-S}\,. \tag{2.6}$$

The integral (2.6) can be performed by means of the localization formula [7, 10]

$$\int_M e^{-S} = \sum_{\phi_0} \frac{e^{-S_0}}{\mathrm{Sdet}^{\frac{1}{2}}\,\mathcal{L}_{\phi_0}} \tag{2.7}$$

where the sum is running over the fixed points ϕ_0 (here assumed to be isolated). The operator in the denominator of (2.7) is defined starting from the super vector field Q (the BRST charge) on the ADHM manifold with $\mathcal{L}_{\phi_0} v = [Q, v]|_{\phi_0}$ on a generic element v of the tangent space.

That the integral (2.6) localizes is not surprising since it computes an "index" of a supersymmetric theory. The crucial observation in [5] is that after suitable deformations (vev and gravitational backgrounds)it localizes on a finite point set $\{\phi_0\}$. The action, S, can be deformed using the symmetries of the ADHM manifold. In fact, the symmetries of the action (2.4) are those that leave the constraints (2.2) invariant. They are given by $g = \mathrm{U}(k)\times\mathrm{U}(N)\times\mathrm{SO}(4)_{\epsilon_{1,2}}\times\mathrm{SO}(4)_{\epsilon_{3,4}}$ with $\mathrm{SO}(4)_{\epsilon_{3,4}} \in \mathrm{SO}(6)_{\mathcal{R}}$ the $\mathcal{R}$-symmetry subgroup preserved by the choice of Q. In particular the quartet indicated between parenthesis in (2.6) is a singlet of g and therefore its bosonic and fermionic contributions to the superdeterminant cancel against each other and can be discarded.

The $\mathrm{U}(k)$ invariance can be fixed by choosing ϕ along the $\mathrm{U}(1)^k$ Cartan subgroup

$$\phi = (\phi(1), \phi(2), \ldots, \phi(k))\,.$$

This leads to an additional jacobian factor $\prod_{s<s'}(\phi(s)-\phi(s'))$: the Vandermonde determinant. The determinant can be exponentiated by introducing an additional BRST pair of auxiliary fields

$$Q\chi_{\mathbb{C}} = H_{\mathbb{C}} \qquad QH_{\mathbb{C}} = [\phi, \chi_{\mathbb{C}}]\,. \tag{2.8}$$

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The old set of auxiliary fields in (2.3) can be extended to include those in (2.8). We denote the new set by a hat

$$\hat{\vec{\chi}} = (\chi_{\mathbb{C}}, \chi_{[\ell_1\ell_2]}, \mu_{\dot{a}}) \qquad \hat{H} = Q\,\hat{\vec{\chi}}\,. \tag{2.9}$$

The action and D-instanton partition function are again given by (2.4) and (2.6) but now in terms of hatted fields with $\mathcal{E}_{\mathbb{C}} = 0$. Indeed the action is quadratic in $\chi_{\mathbb{C}}, H_{\mathbb{C}}$ and after integrating them out the Vandermonde determinant is reproduced. Notice that after the inclusion of $\chi_{\mathbb{C}}$, the number of fermionic degrees of freedom in $\hat{\vec{\chi}}$ matches that of the bosonic ones in $\vec{m}$.

The BRST transformations (2.1) can be deformed to take into account the action of g. For the purposes of the present paper we will consider the symmetries given by the elements in the Cartan of g i.e. $g \in \mathrm{U}(1)^k \times \mathrm{U}(1)^N \times \mathrm{U}(1)^2_{\epsilon_{1,2}} \times \mathrm{U}(1)^2_{\epsilon_{3,4}}$ with g parameterized by $\phi(s), a_\alpha, \epsilon_\ell$ with $s = 1, \ldots, k$, $\alpha = 1, \ldots, N$, $\ell = 1, \ldots, 4$. To this end it is convenient to introduce the auxiliary k and N dimensional spaces V, W, transforming in the fundamental of $\mathrm{U}(k)$, $\mathrm{U}(N)$ and four one dimensional spaces Q_ℓ carrying $\mathrm{U}(1)_\ell$ charges.

$$\begin{aligned}
g\,Q_\ell &= T_\ell\, Q_\ell & \qquad T_\ell &= e^{i\epsilon_\ell} & \qquad \epsilon_\ell &= (\epsilon_1, \epsilon_2, m, -m-\epsilon) \\
g\,V &= T_\phi V & T_\phi &= e^{i\phi} & & \\
g\,W &= T_a W & T_a &= e^{ia} & &
\end{aligned} \tag{2.10}$$

with $\epsilon = \epsilon_1 + \epsilon_2$. The deformed BRST transformations now squares to the infinitesimal transformation $\delta_{\phi,a,\epsilon_\ell} \in g$ given by

$$\begin{aligned}
\delta_{\phi,a,\epsilon_\ell} Q_\ell &= \epsilon_\ell\, Q_\ell & \qquad \epsilon_\ell &= (\epsilon_1, \epsilon_2, m, -m-\epsilon)[2pt] \\
\delta_{\phi,a,\epsilon_\ell} V &= [\phi, V] & \qquad \delta_{\phi,a,\epsilon_\ell} W &= aW\,.
\end{aligned} \tag{2.11}$$

The transformations properties of the fields with respect to g can then be compactly summarized by

$$\begin{aligned}
B_\ell :&\ \ V^* \times V \times (Q_\epsilon + Q_m) \\
\bar{B}_\ell :&\ \ V^* \times V \times \left[\Lambda^2 Q_\epsilon \times Q_m + \Lambda^2 Q_m \times Q_\epsilon\right] \\
\chi_{\mathbb{C}}, \chi_{[\ell_1\ell_2]}, \bar{\chi}_{\mathbb{C}} :&\ \ -V \times V^* \times \left[1 + \Lambda^2(Q_\epsilon + Q_m) + \Lambda^4(Q_\epsilon + Q_m)\right] \\
w_{\dot{\alpha}}, \bar{w}_{\dot{\alpha}} :&\ \ (V \times W^* + V^* \times W \times \Lambda^2 Q_\epsilon) \times (1 + \Lambda^2 Q_m) \\
\mu_{\dot{a}}, \bar{\mu}_{\dot{a}} :&\ \ -(V \times W^* + V^* \times W \times \Lambda^2 Q_\epsilon) \times Q_m
\end{aligned} \tag{2.12}$$

with

$$Q_\epsilon = Q_1 + Q_2 \qquad Q_m = Q_3 + Q_4 \tag{2.13}$$

where Q_ϵ, Q_m are two doublets which are associated to the contributions to the moduli space of the $\mathcal{N} = 2$ gauge vector multiplet and to the matter part respectively.[6] The signs account for the right spin statistics. Λ^p is the space of antisymmetrized p-vectors such that $\Lambda^2 Q_\epsilon = Q_1 Q_2$, $\Lambda^2 Q_m = Q_3 Q_4$ and $\Lambda^4(Q_\epsilon + Q_m) = Q_1 Q_2 Q_3 Q_4$.

[6]If we would neglect $B_{3,4}$, $\vec{m}$ is the moduli space of $\mathcal{N} = 2$.

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The tangent of the moduli space $\vec{m}, \hat{\vec{\chi}}$ is given by collecting all contributions from (2.12)

$$\begin{aligned} T_{\mathcal{M}} &= [V \times V^*(-1 + Q_\epsilon - \Lambda^2 Q_\epsilon) + V \times W^* + V^* \times W \times \Lambda^2 Q_\epsilon](1 - Q_m + \Lambda^2 Q_m) \\ &= [V \times V^*(-1 + Q_\epsilon - \Lambda^2 Q_\epsilon) + V \times W^* + V^* \times W \times \Lambda^2 Q_\epsilon](1 - T_m) + \text{h.c.} \end{aligned} \tag{2.14}$$

with $T_m = e^{im}$. The mathematical origin of (2.14) is explained in appendix C of [7]. Here we give a more intuitive argument. All the terms in squared brackets are the contribution of the $\mathcal{N} = 2$ sector of $\mathcal{N} = 4$. The ADHM constraints and U(k) invariance are represented by the negative terms on the r.h.s. of (2.14) and they are $2k^2 = 2(\dim V)^2$ complex conditions.[7] The ambient space $\mathbb{C}^{2kN+2k^2}$ is given by the positive terms inside the squared brackets. The total tangent space is thus a $2kN$-complex dimensional hypersurface embedded in $\mathbb{C}^{2kN+2k^2}$. Terms proportional to T_m correspond to the contribution of the matter hypermultiplet in $\mathcal{N} = 2^*$. They are a copy of the $\mathcal{N} = 2$ terms but with the opposite spin statistics (as it can be seen already in (2.1)).

The physical content of the localization formula can be now made precise by noticing that the result (2.7) follows by first expanding the action (2.4) up to quadratic order in the fields around a vacuum characterized by ϕ_0 and then performing the gaussian integrations. To this order we can indeed set $\mathcal{E} = 0$ and the action becomes quadratic. The vacua are defined by the critical point equations

$$Q^2 \vec{m} = \delta_{\phi,a,\epsilon} \vec{m} = 0$$

with $\hat{\vec{H}} = \hat{\vec{\chi}} = \vec{\Psi} = 0$. Notice that after having diagonalized the field ϕ and having performed the gaussian integration over the remaining fields, we are left with a complex integral whose poles are given by the fixed point equations. Therefore the sum over ϕ_0 in (2.7) computes the residue. In components

$$\begin{aligned} \delta_{\phi,a,\epsilon} B_{\ell,ss'} &= (\phi(s) - \phi(s') + \epsilon_\ell)\, B_{\ell,ss'} = 0 \qquad \ell = 1,2 \\ \delta_{\phi,a,\epsilon} I &= (\phi(s) - a_\alpha)\, I_{s\alpha} = 0 \\ \delta_{\phi,a,\epsilon} J &= (-\phi(s) + a_\alpha + \epsilon)\, J_{\alpha s} = 0 \end{aligned} \tag{2.15}$$

with $B_3 = B_4 = 0$.

The solutions of (2.15) can be put in one to one correspondence with a set of N Young tableaux $(Y_1, \ldots, Y_N)$ with $k = \sum_\alpha k_\alpha$ boxes distributed between the Y_α's. The boxes in a Y_α diagram are labelled either by the instanton index s running over all boxes $s \in Y_\alpha = 1, \ldots, k_\alpha$ or by a pair of integers i_α, j_α denoting the horizontal and vertical position respectively in the Young diagram.

[7] Moduli spaces are constructed as quotient spaces. One starts from a flat space to later impose some constraints and to mod out by a symmetry group. For the $\mathcal{N} = 2$ case the number of constraints is usually taken to be $3k^2$ (one real and 1 complex) and the symmetry group U(k). After the imposition of the constraints the manifold which is obtained is not complex anymore since its dimension is odd. After modding out the U(k) symmetries the final manifold (the moduli space) is complex. To avoid the real constraint to break the complex structure in intermediate steps, it can be omitted altogether at the price of modifying the group of symmetries of the construction to be $GL(k)$ [25]. This is the case here.

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For simplicity we will set from now on $\epsilon_1 = -\epsilon_2 = \hbar$. This corresponds to consider self-dual gravitational backgrounds. The explicit solutions to (2.15) can then be written as

$$\phi(s \in Y_\alpha) = \phi_{i_\alpha j_\alpha} = a_\alpha + (i_\alpha - 1)\epsilon_1 + (j_\alpha - 1)\epsilon_2 = a_\alpha + (i_\alpha - j_\alpha)\hbar \tag{2.16}$$

and $J = B_\ell = I = 0$ except for the components $B_{1(i_\alpha j_\alpha),(i_\alpha+1 j_\alpha)}$, $B_{2(i_\alpha j_\alpha),(i_\alpha j_\alpha+1)}$, $I_{\alpha,11}$ $\alpha = 1,\ldots,N$. At the critical points the spaces V, W, Q_ℓ become $T_\hbar = e^{i\hbar}$, $T_m = e^{im}$ and $T_{a_\alpha} = e^{ia_\alpha}$ modules allowing the decomposition

$$\begin{aligned} V &= \sum_{s\in Y_\alpha} e^{i\phi(s)} = \sum_{s\in Y_\alpha} T_{a_\alpha} T_\hbar^{i_\alpha - j_\alpha} \qquad W = \sum_{\alpha=1}^{N} T_{a_\alpha} \\ Q_1 &= Q_2^* = T_\hbar \qquad Q_3 = Q_4^* = T_m \,. \end{aligned} \tag{2.17}$$

Plugging (2.17) in (2.14) one finds after a long but straightforward algebra the character

$$\chi = (2 - T_m - T_m^{-1}) \left(\sum_{\alpha,\beta=1}^{N} \sum_{s\in Y_\alpha} T_{E_{\alpha\beta}(s)} + \text{h.c.} \right) \tag{2.18}$$

with $T_{E_{\alpha\beta}(s)} \equiv e^{iE_{\alpha\beta}(s)}$ given in terms of

$$E_{\alpha\beta}(s) = a_{\alpha\beta} + \hbar\, \ell_{\alpha\beta}(s) \tag{2.19}$$

and $\ell_{\alpha\beta}(s)$ is the "length of the hook stretched between the tableaux Y_α, Y_β" and centered on the box $s \in Y_\alpha$. More precisely $\ell_{\alpha\beta}(s)$ is the standard hook length for a box $s \in Y_\alpha$ (the number of white circles in figure 1) plus the vertical difference $\nu_\beta(s) - \nu_\alpha(s)$ between the top ends of the columns of Y_α, Y_β to which s belongs to (the number of black circles in figure 1). Notice that when $s \in Y_\alpha$ lies outside of Y_β, the hook length can be negative since the vertical distance between the two tableaux can be negative. In particular if s lies outside on the right of diagram Y_β then $\nu_\beta = 0$. See appendix C of [7] for a more detailed explanation of the mathematical meaning of (2.18).

The exponents in (2.18) are the eigenvalues of the operator $\mathcal{L}_{\phi_0}$ which enter our localization formula (2.7). The partition function is then given by replacing the sum $\chi = \sum_j e^{i\lambda_j}$ in (2.18) by a product over the eigenvalues λ_j leading to

$$\begin{aligned} Z_k &= \sum_{\phi_0} \frac{1}{\text{Sdet}^{1/2}\mathcal{L}_{\phi_0}} = \sum_{Y=\{Y_\alpha\}} \mathcal{Z}_Y \\ \mathcal{Z}_Y &= \prod_{\alpha,\beta=1}^{n} \prod_{s\in Y_\alpha} \frac{(E_{\alpha\beta}(s)+m)(E_{\alpha\beta}(s)-m)}{E_{\alpha\beta}(s)^2} \end{aligned} \tag{2.20}$$

with $k = |Y|$ the instanton number.

It is convenient to introduce the notation

$$\begin{aligned} f(x) &= \frac{(x-m)(x+m)}{x^2} \\ S_\alpha(x) &= \prod_{\beta\neq\alpha} \left[\frac{(x-a_\beta+m)(x-a_\beta-m)}{(x-a_\beta)^2} \right]. \end{aligned} \tag{2.21}$$

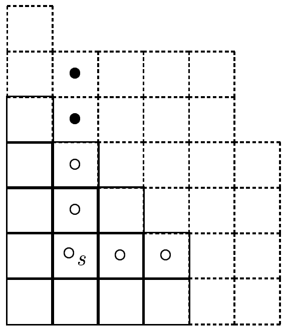

Figure 1: Two generic Young diagrams denoted by the indices α, β in the main text. In the figures the diagram Y_α where the box "s" belongs to is always displayed with solid lines. The hook starts on box "s" a run horizontally till the end of diagram Y_α and vertically till the top end of Y_β. The two tableaux are depicted on top of each other so that in the picture in the left side the solid diagram Y_α contain both solid and dashed boxes.

In terms of these functions the contributions coming from the tableaux with $k = 1, 2$ can be written as [7]

$$
\begin{aligned}
\mathcal{Z}_{\square} &= q \sum_\alpha f(\hbar) S_\alpha(a_\alpha) \\
\mathcal{Z}_{\square,\square} &= \frac{1}{2} q^2 \sum_{\alpha \neq \beta} S_\alpha(a_\alpha) S_\beta(a_\beta)\, f(a_{\alpha\beta} + \hbar) f(a_{\alpha\beta} - \hbar) \frac{f(\hbar)^2}{f(a_{\alpha\beta})^2} \\
\mathcal{Z}_{\square\square} &= q^2 \sum_\alpha f(\hbar) f(2\hbar) S_\alpha(a_\alpha) S_\alpha(a_\alpha + h)
\end{aligned}
\tag{2.22}
$$

with $S_\alpha = S_\alpha(a_\alpha)$ and $\mathcal{Z}_{\substack{\square\\\square}}$ given by $\mathcal{Z}_{\square\square}$ with $\hbar \to -\hbar$. In a similar way the k-instanton partition function can be written in terms of a products of k $S_\alpha(x)$'s.

The multi instanton partition function $Z(q) = \sum_k Z_k q^k$ determines the prepotential $\mathcal{F}(q) = \sum_k \mathcal{F}_k q^k$ via the relation

$$
\mathcal{F}(q) \equiv \lim_{\hbar \to 0} \mathcal{F}(q, \hbar) = \lim_{\hbar \to 0} \hbar^2 ln\, Z(q)\,. \tag{2.23}
$$

The general function $\mathcal{F}(q, \hbar)$ encodes the gravitational corrections to the $\mathcal{N} = 2^*$ superpotential.

3. Quiver gauge theories

3.1 Generalities

Quiver gauge theories are defined by projecting $\mathcal{N} = 4$ SYM with the orbifold group Γ now embedded in the $\mathcal{R}$-symmetry group. This group has been already used to introduce mass deformations parameterized by a $\mathrm{U}(1)_m$ (whose action is given by T_m) breaking the original $\mathrm{SO}(6)$ $\mathcal{R}$-symmetry group of $\mathcal{N} = 4$ down to that of $\mathcal{N} = 2^*$ [16, 15]. Under the action of $\Gamma = \mathbb{Z}_p$ the spaces V, W decompose as

$$V = \sum_{q=0}^{p-1} V_q \qquad W = \sum_{q=0}^{p-1} W_q \tag{3.1}$$

with q labelling the q^{th} irreducible representation R_q of $\mathbb{Z}_p$ under which the corresponding $k_q = \mathrm{dim} V_q$ D(-1) instantons and $N_q = \mathrm{dim} W_q$ D3-branes, transform. The symmetry group $\mathrm{U}(k) \times \mathrm{U}(N)$ becomes $\prod_{q=0}^{p-1} \mathrm{U}(k_q) \times \mathrm{U}(N_q)$.

The orbifold group generator in $\Gamma = Z_p$ is taken along the $\mathrm{U}(1)_m$

$$\mathbb{Z}_p \quad : m \to m + \frac{2\pi}{p} \qquad a_\alpha \to a_\alpha + \frac{2\pi q_\alpha}{p} \tag{3.2}$$

q_α, $\alpha = 1, \ldots, N$ can take integer values between 0 and $p-1$, specifying the representations under which the α^{th} D3-brane transforms. In particular the integers N_q characterizing the unbroken gauge group $\prod_q \mathrm{U}(N_q)$ are given by the number of times that q appears in $(q_1, q_2, \ldots q_N)$. The integers $k_q = \mathrm{dim} V_q$ are given by the total number of boxes in the Young tableaux with $q_\alpha = q$.

Equivalently in terms of the decompositions (3.1) the orbifold group action can be written as

$$\mathbb{Z}_p \quad : m \to m + \frac{2\pi}{p} \qquad V_q \to e^{\frac{2i\pi q}{p}} V_q \qquad W_q \to e^{\frac{2i\pi q}{p}} W_q \,. \tag{3.3}$$

The Γ invariant component of the tangent space (2.14) under (3.3) can then be written as

$$\begin{aligned} T_\Gamma = & \sum_{q=0}^{p-1} \left[V_q^* V_q (Q_\epsilon - 1 - \Lambda^2 Q_\epsilon) + W_q^* V_q + V_q^* W_q \Lambda^2 Q_\epsilon\right] - \\ & - T_m \sum_{q=0}^{p-1} \left[V_q^* V_{q-1} (Q_\epsilon - 1 - \Lambda^2 Q_\epsilon) + W_q^* V_{q-1} + V_q^* W_{q-1} \Lambda^2 Q_\epsilon\right] + \mathrm{h.c.} \,. \end{aligned} \tag{3.4}$$

In particular for $V = V_0, V_{q \geq 1} = 0$, that is when all the instantons sit in the $q = 0$ D3-branes, the Γ-invariant moduli space (3.4) reduces to

$$\begin{aligned} T_\Gamma = & \left[V_0^* V_0 + V_0^* V_0 - V_0^* V_0 \Lambda^2 Q_\epsilon - V_0^* V_0 + W_0^* V_0 + V_0^* W_0 \Lambda^2 Q_\epsilon\right] - \\ & - Q_m \left[W_{N-1}^* V_0 + V_0^* W_1 \Lambda^2 Q_\epsilon\right] \end{aligned}$$

which coincides with the tangent to the moduli space of an $\mathcal{N} = 2$ SYM theory with gauge group $\mathrm{U}(N_0)$, N_1 fundamentals and N_{p-1} anti-fundamentals (see (3.21) in [7]). In particular for $p = 2$ and $N_0 = N_1$ we find the $\mathcal{N} = 2$ $\mathrm{U}(N_0)$ superconformal theory with $2N_0$ fundamental matter [7]. The general case described in (3.4) corresponds to $\mathcal{N} = 2$ gauge theories with product gauge groups $\prod_q \mathrm{U}(n_q)$ and bifundamental matter. The character of the Γ-invariant tangent space follows from the Γ-invariant component of (2.18) under (3.2) and can be written as

$$\chi_{m,\Gamma} = \sum_{\alpha,\beta}^{N} \sum_{s \in Y_\alpha} T_{E_{\alpha\beta}(s)} \left(2\, \delta_{q_\alpha, q_\beta} - T_m\, \delta_{q_\alpha, q_{\beta+1}} - T_m^{-1}\, \delta_{q_\alpha, q_{\beta-1}}\right) + \mathrm{h.c.} \tag{3.5}$$

with $q = 0, \ldots, p-1$ running over the various factors in the product gauge group $\prod_q \mathrm{U}(n_q)$.

The partition function reads

$$Z_k = \sum_{\{Y_\alpha\}} \prod_{\alpha_q,\beta_q} \prod_{s\in Y_\alpha} \frac{(E_{\alpha_q\beta_{q+1}}(s)+m)(E_{\alpha_q\beta_{q-1}}(s)-m)}{E_{\alpha_q\beta_q}(s)^2}\,. \tag{3.6}$$

Notice in particular that the contribution of the matter fields (m-dependent eigenvalues) turns out to always be off-diagonal in agreement with the fact that they always transform in bifundamental representations.

3.2 The quiver prepotential

Finally we derive the multi instanton partition functions and prepotentials describing the low energy physics on quiver gauge theories descending from $\mathcal{N}=2^*$ gauge theories via $\mathbb{Z}_p$ projections.[8]

For concreteness we restrict ourselves to the case $\mathrm{U}(N_0)\times\mathrm{U}(N_1)$, i.e. $N_q=0$ for $q>1$. More precisely we consider a $\mathcal{N}=2$ quiver theory with gauge group $\mathrm{U}(N_0)\times\mathrm{U}(N_1)$ and bifundamental hypermultiplets with mass m in representations $(N_0,\bar{N}_1)$ and $(N_{p-1},\bar{N}_0)$ of the gauge group. Notice that $R_1=\bar{R}_1$ is real for $\mathbb{Z}_2$ but complex for $\mathbb{Z}_p$ with $p>2$. This implies in particular that matter in the fundamental and antifundamental will transform in the same way under $\mathbb{Z}_p$ for $p=2$ and therefore both fundamental and anti-fundamental matter will survive the $\mathbb{Z}_2$ projection. On the other hand for $\mathbb{Z}_p$ projections with $p\geq 2$ only the fundamental matter survives the orbifold projection.

We start by consider the $\mathbb{Z}_2$ case. It is convenient to split the gauge index α into $\alpha=r,\hat{r}$ with $r=1,\ldots,N_0$ and $\hat{r}=1,\ldots,N_1$. This corresponds to choose two sets of D3-branes transforming in the R_0 and R_1 representation of $\mathbb{Z}_2$.

We now need to project (2.21) onto its invariant components. The projection is given by (3.2) with $q_r=0, q_{\hat{r}}=1$. Given the gauge group $\mathrm{U}(N_0)\times\mathrm{U}(N_1)$ we see that under (3.2) the combinations $a_{r_1r_2}, a_{\hat{r}_1,\hat{r}_2}, a_{r_1,\hat{r}_2}\pm m$ are invariant. The invariant components in (2.21) are then given by

$$\begin{aligned}
f(a_{r_1\hat{r}_2}+x) &= (a_{r_1\hat{r}_2}+x)^2-m^2 & f(x) &= \frac{1}{x^2}\\
f(a_{r_1r_2}+x) &= \frac{1}{(a_{r_1r_2}+x)^2} & f(a_{\hat{r}_1\hat{r}_2}+x) &= \frac{1}{(a_{\hat{r}_1\hat{r}_2}+x)^2}\\
S_{r_1}(x) &= \frac{\prod_{\hat{r}_2}(x-a_{\hat{r}_2})^2-m^2}{\prod_{r_2\neq r_1}(x-a_{r_2})^2} & S_{\hat{r}_1}(x) &= \frac{\prod_{r_2}(x-a_{r_2})^2-m^2}{\prod_{\hat{r}_2\neq\hat{r}_1}(x-a_{\hat{r}_2})^2}\,.
\end{aligned} \tag{3.7}$$

In (3.7) the variable x does not transform under (3.2). Once again one can verify the formulae we obtain for $S_{r_1}(x)$, $S_{\hat{r}_1}(x)$ against those coming from the Seiberg and Witten curves [19]. As expected, the contributions coming from the massless $\mathcal{N}=2$ vector multiplet appear in the denominators of (3.7) and are in the adjoint of $\mathrm{U}(N_0)\times\mathrm{U}(N_1)$ (diagonal terms). The contributions from the massive matter are in the numerators and appear off-diagonally as bifundamentals. This can also be seen from the character (3.4)

[8] In this section we will give the lowest terms in the expansion of the prepotential using an analytical method for pedagogical purposes. To check highest order terms, we use a Mathematica code.

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where the term $-T_m$, representing the contribution of the massive hypermultiplet, is the only term transforming under the orbifold projection (3.2). To compensate this action the corresponding bilinear in V_p, W_q should transform accordingly.

The partition function is given again by (2.22) but now with α running over $r, \hat{r}$ and $f(x), S_\alpha(x)$ given by (3.7). The prepotential is given by (2.23) leading to

$$\begin{aligned}\mathcal{F} = {} & q\sum_r S_r + \hat{q}\sum_{\hat{r}} S_{\hat{r}} + q^2\left(\frac{1}{4}\sum_r S_r S_r'' + \sum_{r_1\neq r_2}\frac{1}{a^2_{r_1 r_2}} S_{r_1}S_{r_2}\right) + \\ & +\hat{q}^2\left(\frac{1}{4}\sum_{\hat{r}} S_{\hat{r}} S_{\hat{r}}'' + \sum_{\hat{r}_1\neq \hat{r}_2}\frac{1}{a^2_{\hat{r}_1 \hat{r}_2}} S_{\hat{r}_1}S_{\hat{r}_2}\right) - \\ & -2q\hat{q}\sum_{r_1,\hat{r_2}}\frac{(a_{r_1\hat{r}_2}+\hbar)^2+m^2}{(a^2_{r_1\hat{r}_2}-m^2)^2} S_r S_{\hat{s}} + \cdots \end{aligned} \tag{3.8}$$

where $S_r = S_r(a_r)$. Here we weight the instanton contributions by $q = e^{2\pi i k\tau}$, $\hat{q} = e^{2\pi i\hat{k}\hat{\tau}}$ with $\tau, \hat{\tau}$ the coupling constants of the two gauge groups.

In appendix B we present the results for Z_p quivers with $p > 1$ corresponding to quiver gauge theories with a single hypermultiplet in the bifundamental.

Finally one can consider quivers of U(N) rather than SU(N) groups. As an illustration consider the $\mathbb{Z}_2$-quiver of $\mathcal{N} = 2^*$ SYM with starting gauge group U(2). There are two possibilities according to whether we choose $q_1 = q_2$ or $q_1 \neq q_2$ in (3.2). They lead to quiver gauge theories with gauge groups U(2) and U(1)^2 respectively. In the former case the projection (3.2) remove precisely the massive fields leading to pure $\mathcal{N} = 2$ SYM. In the latter case (with $q_1 = 0, q_2 = 1$) one finds

$$\begin{aligned} Z(q,\hbar) = {} & 1+\left(\frac{q}{\hbar^2}\right)(8a^2-2m^2)+\left(\frac{q}{\hbar^2}\right)^2(32a^4+2m^4-3m^2\hbar^2+\hbar^4-16a^2m^2-12a^2\hbar^2)+ \\ & +\frac{2}{3}\left(\frac{q}{\hbar^2}\right)^3(4a^2-m^2)(32a^4+2m^4-9m^2\hbar^2+7\hbar^4-16a^2m^2-12\hbar^2a^2)+ \\ & +\frac{1}{6}\left(\frac{q}{\hbar^2}\right)^4(1024a^8+4m^8-36m^6\hbar^2+71m^4\hbar^4-39m^2\hbar^6+12\hbar^8-1024a^6m^2- \\ & \quad -768a^6\hbar^2+384a^4m^4-392a^4m^2\hbar^2+752a^4\hbar^4-64a^2m^6+ \\ & \quad +240a^2m^4\hbar^2-280a^2m^2\hbar^4-84a^2\hbar^6)+\cdots \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \mathcal{F}(q) = {} & \Bigg[(-8a^2+2m^2)q+(4a^2+3m^2)q^2+\frac{8}{3}(-4a^2+m^2)q^3+ \\ & +\left(10a^2+\frac{7m^2}{2}\right)q^4+\frac{12}{5}(-4a^2+m^2)q^5+\frac{4}{3}(4a^2+3m^2)q^6+O(q^7)\Bigg]- \\ & -\hbar^2\left[q^2+\frac{3}{2}q^4+\frac{4}{3}q^6+O(q^7)\right]. \end{aligned} \tag{3.10}$$

Remarkably the expansion in $\hbar$ stops now at $\hbar^2$ i.e. instanton corrections to gravitational terms $\mathcal{F}_g\hbar^{2g}$ vanish beyond one loop $g \geq 1$. This is clearly peculiar for this simple situation where only powers of $\hbar$ are left in the denominator in (3.9) due to the orbifold projection. This implies in particular that $\mathcal{F}_k(a, m, \hbar)$ is given by a homogenous polynomial of $a, m, \hbar$ of order two.

4. Orientifolds

SYM theories with gauge group SO/Sp can be studied by including an O3 orientifold plane in the D3-D(-1) system. The orientifold plane can be thought of as a mirror where the open strings reflect, losing their orientation. Like in the orbifold case the character of the tangent space to the multi-instanton moduli manifold can be found via a projection. The projection operator is $(1 \mp \Omega\mathbb{Z}_2)/2$. Besides a reflection of the six transverse coordinates to the O3 plane, we have a conjugation (worldsheet parity) on the Chan-Paton indices. In terms of the previously introduced V, W and Q_ϵ spaces, we have

$$\Omega\mathbb{Z}_2 : V \to V^* \qquad W_\alpha \to W_\alpha^* \qquad Q_\epsilon \to -Q_\epsilon \,. \tag{4.1}$$

The choices $\mp\Omega\mathbb{Z}_2$ correspond to SO / Sp gauge theories. The gauge symmetries of the D(-1)-D3-O3 system become SO(N) $\times$ Sp($2k$) or Sp($2n$) $\times$ SO(K). After the v.e.v.'s are turned on, the D3-branes distribute over the ϕ-plane in mirror-image pairs $a_\alpha = a_1, \dots a_n, -a_1, \dots, -a_n$ for SO($2n$)/ Sp($2n$) while for SO($2n+1$) an additional D3-brane sits at the origin (($a_{2n+1} = 0$) on top of the orientifold plane. In a similar way instantons distribute symmetrically between the two sides of the O3 mirror.[9] This corresponds to distributing instantons over a particular set of Young tableaux of the type

$$\begin{aligned} Y_{\mathrm{SO}(2n)} &= (Y_1, \dots, Y_n, Y_1^T, \dots, Y_n^T) \\ Y_{\mathrm{SO}(2n+1)} &= (Y_1, \dots, Y_n, Y_1^T, \dots, Y_n^T, \emptyset) \\ Y_{\mathrm{Sp}(2n)} &= (Y_1, \dots, Y_n, Y_1^T, \dots, Y_n^T, Y_\nu) \,. \end{aligned} \tag{4.2}$$

As it is stressed in figure 2, the second half of the Young tableaux are conjugate to the first half i.e. the number of boxes in rows and columns are interchanged. The Sp($2n$) gauge theory admits special solutions to (2.15) which are not present in the parent SU($2n$) gauge theory: we call these solutions "fractional instantons". They are distributed symmetrically around the O3-plane at the origin and outside of any D3-brane. Also these solutions can be put in relation with particular types of tableaux Y_ν (see [20] for details). We will start by considering the case $Y_\nu = \emptyset$ and include the contributions of fractional instantons only at the end.

The choice (4.2) implies that the spaces V and W are $2k$ and $2n$ dimensional respectively and real and therefore the projection under $(1 \mp \Omega\mathbb{Z}_2)/2$ makes sense. More precisely

$$\begin{aligned} V &= \sum_{\alpha=1}^{2n} \sum_{s \in Y_\alpha} e^{i\phi(s)} = \sum_{r=1}^{n} \sum_{s \in Y_r} e^{i\phi(s)} + \sum_{r=1}^{n} \sum_{s \in Y_{\hat r}} e^{i\phi(s)} = V_\phi + V_\phi^* \\ W &= \sum_{\alpha=1}^{2n} T_{a_\alpha} = \sum_{r=1}^{n} e^{ia_r} + \sum_{\hat r=1}^{n} e^{ia_{\hat r}} = W_+ + W_+^* \,. \end{aligned} \tag{4.3}$$

[9] No D(-1)-branes are allowed to be superposed to the $(2n+1)^{th}$ D3-brane in the SO($2n+1$) case.

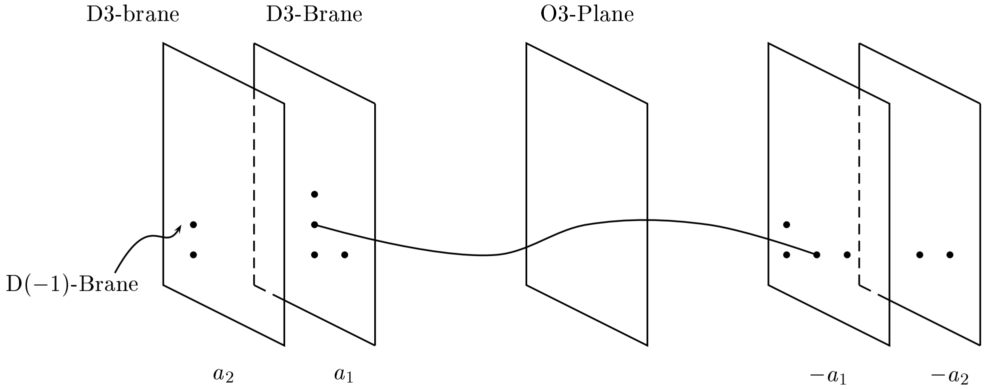

Figure 2: D(−1)-D3 branes in the presence of an orientifold plane. We have also drawn a D(−1)-D(−1) open string. See the text for a more detailed explanation of the figure.

Non-diagonal terms like $V \times W$ or $V_{\phi(s)} \times V_{\phi(s')}$ with $s \neq s'$ come always in $\pm$ pairs and therefore only one half of them survive the projection. The $(1 \mp \Omega\mathbb{Z}_2)/2$ invariant tangent space can then be written by applying the projection operator to (2.14) (in which the V, W spaces are those in (4.3)) and taking the trace. The term $\frac{1}{2}\mathrm{Tr}\,1$ (the annulus) gives half the contribution of SU$(2n)$ evaluated on the symmetric configurations (4.2) and can be computed using (2.18). On the other hand the trace $\frac{1}{2}\mathrm{Tr}\Omega\mathbb{Z}_2$ (Moebius strip)[10] receives only contributions from diagonal terms since the contributions of $\pm$ paired states cancel against each other. Collecting the two contribtuions for the pure $\mathcal{N} = 2$ SYM case one finds

$$T_{\mathcal{M}} = \left[\frac{1}{2}(V^2 \mp V_{2\phi}) \times Q_\epsilon - \frac{1}{2}(V^2 \pm V_{2\phi}) \times (1 + \Lambda^2 Q_\epsilon) + \frac{1}{2} V \times W \times (1 + \Lambda^2 Q_\epsilon)\right] + \text{h.c.} \quad (4.4)$$

with upper/lowers signs corresponding from now on to SO(N)/Sp(N) gauge groups respectively and

$$\begin{aligned} V_{2\phi} &= \sum_{\alpha=1}^{2n} \sum_{s \in Y_\alpha} e^{2i\phi(s)} = \sum_{r=1}^{n} \sum_{s \in Y_r} \left(e^{2i\phi(s)} + e^{-2i\phi(s)}\right) \\ \phi(s \in Y_\alpha) &= a_\alpha + (i_\alpha - j_\alpha)\hbar \end{aligned} \quad (4.5)$$

For instance, specifying to SO(N), the ADHM constraints (the terms multiplied by $\Lambda^2 Q_\epsilon$) correspond to matrices in the symmetric $k(2k+1)$ representation (the adjoint of Sp$(2k)$) while the B-moduli (which are the terms multiplied by Q_ϵ) come in the antisymmetric representation of Sp$(2k)$ as expected. This is in agreement with the standard ADHM construction for SO(N) groups. The Sp(N) case behaves analogously with the symmetric and antisymmetric representations exchanged.

The D-instanton character is given by

$$\chi = \sum_{\alpha=1}^{2n} \sum_{s \in Y_\alpha} \left[\sum_{\beta}^{2n} T_{E_{\alpha\beta}(s)} \mp \frac{1}{2} e^{2i\phi(s)}(2 + T_\hbar + T_\hbar^{-1})\right] + \text{h.c.}. \quad (4.6)$$

[10] To stress the similarity with the construction of the open string partition function, in parenthesis we have given the names of the analogous terms which arise in that case.

The sum over α, β splits into four terms according to whether we choose α, β between the set $r_{1,2} = 1, \dots n \equiv \frac{1}{2}N$ or their images $\hat{r}_{1,2}$. In particular notice that while $E_{r_1 r_2}(s) = -E_{\hat{r_1}\hat{r_2}}$ is a function of the difference $a_{r_1 r_2} \equiv a_{r_1} - a_{r_2}$, the contributions $E_{r_1 \hat{r_2}}(s)$,$E_{r_1 \hat{r_2}}(s)$ coming from mixed pair of tableaux are functions of the sum $a_{r_1 \hat{r_2}} \equiv a_{r_1} + a_{r_2}$. The result can then be written as

$$\chi = \sum_{r_1=1}^{n} \sum_{s \in Y_{r_1}} \left[\sum_{r_2=1}^{n} \left(2\, T_{E_{r_1 r_2}(s)} + T_{E_{r_1 \hat{r_2}}} + T_{E_{\hat{r_1} r_2}} \right) \mp e^{2i\phi(s)} (2 + T_\hbar + T_\hbar^{-1}) \right] + \text{h.c.}. \quad (4.7)$$

Here and in the following we write $\sum_{s \in Y_{r_1}} T_{E_{\hat{r_1} r_2}}$ to indicate $\sum_{s \in Y_{\hat{r}_1}} T_{E_{\hat{r_1} r_2}}$. The diagram Y_α to which the box "s" belongs will be unambiguously specified by the first label of $E_{\alpha\beta}$. Specifying to the SO$(2n)$ case, one then finds

$$\mathcal{Z}_{2k}^{\mathrm{SO}(2n)} = \prod_{r_1=1}^{n} \prod_{s \in Y_{r_1}} \left[4\, \phi(s)^2 \, (4\, \phi(s)^2 - \hbar^2) \prod_{r_2=1}^{n} \frac{1}{E_{r_1 r_2}(s)^2 \, E_{r_1 \hat{r}_2}(s) \, E_{\hat{r}_1 r_2}(s)} \right] \quad (4.8)$$

For SO$(2n+1)$ we have an additional contribution from the strings ending on the brane at the origin, i.e. $\prod_{s \in Y_{r_1}} |E_{r_1, n+1} E_{\hat{r}_1, n+1}| = \prod_{s \in Y_{r_1}} \phi(s)^2$, so that one finds

$$\mathcal{Z}_{2k}^{\mathrm{SO}(2n+1)} = \mathcal{Z}_{2k}^{\mathrm{SO}(2n)} \prod_{r_1=1}^{n} \prod_{s \in Y_{r_1}} \frac{1}{\phi(s)^2} \quad (4.9)$$

To check these results (and those we will find later), we find it convenient to compute the lowest instanton contribution. Specifying to a single instanton pair ($k = 1$) one finds

$$\begin{aligned} Z_2^{\mathrm{SO}(2n)} &= \frac{1}{\hbar^2} \sum_{r_1} 4 a_{r_1}^2 \prod_{r_2 \neq r_1} \frac{1}{(a_{r_1}^2 - a_{r_2}^2)^2} \\ Z_2^{\mathrm{SO}(2n+1)} &= \frac{4}{\hbar^2} \sum_{r_1} \prod_{r_2 \neq r_1} \frac{1}{(a_{r_1}^2 - a_{r_2}^2)^2} \end{aligned} \quad (4.10)$$

(4.10) can be rewritten as

$$Z_2^{\mathrm{SO}(N)} = \frac{1}{\hbar^2} \sum_{r_1=1}^{n} S_{r_1}^{\mathrm{SO}(N)}(a_{r_1})$$

with

$$\begin{aligned} S_{r_1}^{\mathrm{SO}(2n)}(x) &= \frac{(2x)^4}{(x + a_{r_1})^2} \prod_{r_2 \neq r_1} \frac{1}{(x^2 - a_{r_2}^2)^2} \\ S_{r_1}^{\mathrm{SO}(2n+1)}(x) &= \frac{(2x)^4}{x^2 (x + a_{r_1})^2} \prod_{r_2 \neq r_1} \frac{1}{(x^2 - a_{r_2}^2)^2} \, . \end{aligned} \quad (4.11)$$

In a similar way, $\mathcal{Z}_{2k}$ contributions can be written as products of k functions $S_{r_1}^{\mathrm{SO}(2n)}(x)$ (see [20] for details). The appearance of functions of the type of (4.10) is typical in the analysis á la Seiberg-Witten. They are related to the polynomial $S(x)$ entering in the Seiberg-Witten curve via

$$S^G(x) = \frac{S_r^G(x)}{(x-a_r)^2} \tag{4.12}$$

leading to[11]

$$\begin{aligned} S^{\mathrm{SO}(2n)}(x) &= (2x)^4 \prod_{r=1}^{n} \frac{1}{(x^2-a_r^2)^2} \\ S^{\mathrm{SO}(2n+1)}(x) &= \frac{(2x)^4}{x^2} \prod_{r=1}^{n} \frac{1}{(x^2-a_r^2)^2}\,. \end{aligned} \tag{4.13}$$

This can be compared with the results of table 3 of [22][12] (see also [23]). Remarkably the first few instanton contributions contains already enough information to reconstruct from it the full Seiberg-Witten curve. This will be used later on to test our results in the case of massive deformations.

The contributions of regular instantons to $\mathrm{Sp}(2n)$ can be found from the $\mathrm{SO}(2n)$ case by flipping the sign of the $V_{2\phi}$ term in (4.4) leading to

$$\mathcal{Z}_{2k,0}^{\mathrm{Sp}(2n)} = \prod_{r_1=1}^{n} \prod_{s\in Y_{r_1}} \left[\frac{1}{4\,\phi(s)^2\,(4\,\phi(s)^2-\hbar^2)} \prod_{r_2=1}^{n} \frac{1}{E_{r_1r_2}(s)^2\,E_{r_1\hat{r}_2}(s)\,E_{\hat{r}_1r_2}(s)} \right]. \tag{4.14}$$

The second subscript in $\mathcal{Z}_{2k,0}^{\mathrm{Sp}(2n)}$ reminds the reader that only instanton configurations with no fractional instantons ($Y_\nu = \emptyset$) are taken into account. In general we will write

$$Z^{\mathrm{Sp}(2n)}(q) = \sum_K \mathcal{Z}_K^{\mathrm{Sp}(2n)} = \sum_{k,m} \mathcal{Z}_{2k,m}^{\mathrm{Sp}(2n)} q^{2k+m} \tag{4.15}$$

with m counting the number of fractional instantons. Fractional instantons can be included by replacing $V \to V + V_\nu$ in (4.5), with V_ν describing the fractional instanton configuration. The extra contributions to the character (4.5) are given by

$$\delta T_{\mathcal{M}} = \frac{1}{2} V_{2\nu} \times (Q+1+\Lambda^2 Q) + \frac{1}{2}(V_\nu^2 + 2VV_\nu) \times (Q-1-\Lambda^2 Q) + \frac{1}{2} V_\nu \times W \times (1+\Lambda^2 Q)\,. \tag{4.16}$$

A single fractional instanton is constrained to stay at the origin, two must be symmetrically arranged so to be centered at $(x,y) = (\pm\frac{\hbar}{2}, 0)$ or $(x,y) = (0, \pm\frac{\hbar}{2})$, etc. We denote the corresponding V_ν and $V_{2\nu}$ by $V_\bullet$, $V_{\bullet\bullet}$ and $V_{2\bullet}$, $V_{2\bullet\bullet}$ respectively. Explicitly

$$\begin{aligned} V_\bullet &= 1 & V_{2\bullet} &= 1 \\ V_{\bullet\bullet} &= T_{\frac{\hbar}{2}} + T_{-\frac{\hbar}{2}} & V_{2\bullet\bullet} &= T_\hbar + T_{-\hbar}\,. \end{aligned} \tag{4.17}$$

Plugging in the character one finds

$$\begin{aligned} \delta\chi_\bullet &= 2\,T_\hbar + (2\,T_\hbar - 2)\,V + W + \text{h.c.} \\ \delta\chi_{\bullet\bullet} &= 2\,T_\hbar + 2\,T_{2\hbar} + 2\,T_{\frac{\hbar}{2}}\,(T_\hbar - T_{-\hbar})\,V + 2\,T_{\frac{\hbar}{2}}\,W + \text{h.c.} \end{aligned} \tag{4.18}$$

[11] For $\mathrm{SU}(n)$ one finds $S^{\mathrm{SU}(n)}(x) = \prod_{r=1}^{n} \frac{1}{(x-a_r)^2}$.

[12] With respect to table 3 of [22] our results is globally multiplied by 2^{4k}.

and

$$
\begin{aligned}
\mathcal{Z}_{2k,1}^{\mathrm{Sp}(2n)} &= \frac{1}{2}\,\mathcal{Z}_{2k,0}^{\mathrm{Sp}(2n)}\frac{1}{\hbar^2}\prod_{r_1=1}^{n}\left[\frac{1}{(-a_{r_1}^2)}\prod_{s\in Y_{r_1}}\frac{\phi^4(s)}{(\phi^2(s)-\hbar^2)^2}\right]\\
\mathcal{Z}_{2k,2}^{\mathrm{Sp}(2n)} &= \frac{2}{2^2}\,\mathcal{Z}_{2k,0}^{\mathrm{Sp}(2n)}\frac{1}{4\hbar^4}\prod_{r_1=1}^{n}\left[\frac{1}{(a_{r_1}^2-\frac{\hbar^2}{4})^2}\prod_{s\in Y_{r_1}}\frac{(\phi^2(s)-\frac{\hbar^2}{4})^2}{(\phi^2(s)-\frac{9\hbar^2}{4})^2}\right].
\end{aligned}
\tag{4.19}
$$

In (4.19) we weight the contribution of m-fractional instantons by 2^{-m}. The extra factor of two for $m=2$ takes into account the identical contributions coming from the two choices of Y_ν, i.e. $(x,y)=(\pm\frac{\hbar}{2},0)$ or $(x,y)=(0,\pm\frac{\hbar}{2})$. For $k=1,2$ one finds

$$
\begin{aligned}
Z_{0,1}^{\mathrm{Sp}(2n)} &= \frac{1}{2\hbar^2}\prod_{r_1=1}^{n}\frac{1}{(-a_{r_1}^2)}\\
Z_{2,0}^{\mathrm{Sp}(2n)} &= \sum_{r}\frac{1}{\hbar^2}\frac{1}{4a_{r_1}^2(4a_{r_1}^2-\hbar^2)^2}\prod_{r_2\neq r_1}\frac{1}{(a_{r_1}^2-a_{r_2}^2)^2}\\
Z_{0,2}^{\mathrm{Sp}(2n)} &= \frac{1}{8\hbar^4}\prod_{r_1=1}^{n}\frac{1}{(a_{r_1}^2-\frac{\hbar^2}{4})^2}
\end{aligned}
\tag{4.20}
$$

in agreement with formulas (B.4) in [20]. The Seiberg-Witten curve is now given by (table 3 of [22])

$$
S^{\mathrm{Sp}(2n)}(x)=\frac{\bar{S}_0(x)}{(2x)^4}=\frac{1}{(2x)^4\prod_{r=1}^{n}(x^2-a_r^2)^2}\,. \tag{4.21}
$$

Now $\hbar^2 Z_{0,1}^{\mathrm{Sp}(2n)}=\frac{1}{2}\bar{S}_0(0)^{\frac{1}{2}}$ [22].

Formulae (4.8), (4.9), (4.14), (4.19) can then be thought of as a deformation of the Seiberg and Witten curves to account for gravitational backgrounds in the gauge theory.

The character corresponding to the case in which matter transforming in the adjoint representation is added to the lagrangian is obtained by multiplying (4.6) by $\frac{1}{2}(2-T_m-T_m^{-1})$, see (2.18). This term is telling us that the extra contribution due to this matter is given by the inverse of the square root of the pure gauge result shifted by $+m$ times with eigenvalues shifted by $\pm m$. More precisely, the partition functions can then be read from (4.8), (4.9), (4.14), (4.19) by replacing

$$
\begin{aligned}
\frac{1}{E_{\alpha\beta}(s)} &\to \frac{(E_{\alpha\beta}(s)^2-m^2)^{\frac{1}{2}}}{E_{\alpha\beta}(s)}\\
(\phi(s)+x) &\to \left[(\phi(s)+x)^2-m^2\right]^{-\frac{1}{2}}\,.
\end{aligned}
\tag{4.22}
$$

Although is not obvious at first sight, the partition function after the replacements (4.22) does not contain any square roots. In particular applying (4.22) to (4.13) one finds for the massive matter contributions

$$
\begin{aligned}
S_m^{\mathrm{SO}(2n)}(x) &= \frac{\prod_{r=1}^{n}[(x+m)^2-a_r^2][(x-m)^2-a_r^2]}{(2x+m)^2(2x-m)^2}\\
S_m^{\mathrm{SO}(2n+1)}(x) &= \frac{(x+m)(x-m)\prod_{r=1}^{n}[(x+m)^2-a_r^2][(x-m)^2-a_r^2]}{(2x+m)^2(2x-m)^2}\,.
\end{aligned}
\tag{4.23}
$$

The Sp$(2n)$ can be treated in an analogous way starting from (4.21)

$$S_m^{\mathrm{Sp}(2n)}(x) = (4x^2 - m^2)^2 \prod_{r=1}^{n} [(x+m)^2 - a_r^2][(x-m)^2 - a_r^2] \,. \tag{4.24}$$

All of these results are in agreement with table 3 of [22].

The contribution of matter in the fundamental representation, with mass $\pm m_f$, is easily added introducing the additional term

$$\delta T_{\mathcal{M}} = -(V_\phi + V_\phi^*)(T_{m_f} + T_{m_f}^*) \tag{4.25}$$

in (4.4). With respect to the discussion leading to (3.21) in [7] we have also reflected the D7-brane with respect to the O3-plane. The character is modified by a term

$$\delta\chi = -\sum_{r=1}^{n} \sum_{s\in Y_r} \left(e^{i(\phi(s)+m_f)} + e^{i(\phi(s)-m_f)} \right) + \text{h.c.} \tag{4.26}$$

which, in turn, leads to the term

$$Z_{m_f} = \prod_{r=1}^{n} \prod_{s\in Y_r} (\phi^2(s) - m_f^2) \tag{4.27}$$

to be multiplied in the numerator of the partition functions (4.8), (4.9), (4.14). This leads to an additional factor $(x^2 - m_f^2)$ in the numerators of the Seiberg-Witten functions $S(x)$, once again in agreement with table 3 of [22]. After this computation was finished ref. [27] appeared which has an overlap with the results of this section.

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A. $\mathcal{N} = (2,2)$ gauge theory

We use for the $D = 2$ supermultiplets the same notation of their ancestors in $D = 4$. Supersymmetry transformations follow from field redefinitions in formulae (2.12) and (2.14) of [26].[13] The four components of the vector field v_m in $D = 4$ will be written as $v_{\pm\pm} \equiv \frac{1}{2}(v_0 \pm v_3)$, $B = \frac{1}{2}(v_1 - iv_2)$ and $\bar{B} = \frac{1}{2}(v_1 + iv_2)$.

[13] Our conventions: $\delta v_m = i\bar{\epsilon}_{\dot\alpha}\bar{\sigma}_m^{\dot\alpha\alpha}\lambda_\alpha + i\epsilon^\alpha\sigma_{m,\alpha\dot\alpha}\bar{\lambda}^{\dot\alpha}$, $\sigma_m = (1,\vec{\sigma}), \bar{\sigma}_m = (1,-\vec{\sigma})$, $\lambda_\alpha = \begin{pmatrix}\lambda_- \\ \lambda_+\end{pmatrix}$, $\bar{\lambda}^{\dot\alpha} = \begin{pmatrix}\bar{\lambda}^- \\ \bar{\lambda}^+\end{pmatrix}$, $\bar{\epsilon}_{\dot\alpha} = (\bar{\epsilon}_- \; \bar{\epsilon}_+)$, $\epsilon^\alpha = (\epsilon^- \; \epsilon^+)$. In addition we make the replacements: $\phi_s \to i\sqrt{2}\,\Phi_s$, $F_s \to \frac{i}{\sqrt{2}} H_s$ in the chiral multiplets.

Vector Multiplet:

$$
\begin{aligned}
\delta v_{++} &= i\,\bar\epsilon_+\lambda_+ + i\,\epsilon_+\bar\lambda_+ \\
\delta v_{--} &= i\,\bar\epsilon_-\lambda_- + i\,\epsilon_-\bar\lambda_- \\
\delta B &= -i\,\bar\epsilon_+\lambda_- - i\,\epsilon_-\bar\lambda_+ \\
\delta \bar B &= -i\bar\epsilon_-\lambda_+ - i\,\epsilon_+\bar\lambda_- \\
\delta\lambda_+ &= i\epsilon_+\,D - 2\,\epsilon_+\,F_{-+} + 4\,\epsilon_-\nabla_{++}\bar B \\
\delta\lambda_- &= i\,\epsilon_-\,D + 2\,\epsilon_-\,F_{-+} + 4\,\epsilon_+\nabla_{--}B \\
\delta\bar\lambda_+ &= -i\,\bar\epsilon_+\,D - 2\,\bar\epsilon_+\,F_{-+} + 4\,\bar\epsilon_-\nabla_{++}B \\
\delta\bar\lambda_- &= -i\bar\epsilon_-\,D + 2\,\bar\epsilon_-\,F_{-+} + 4\,\bar\epsilon_+\nabla_{--}\bar B \\
\delta D &= -2\,\bar\epsilon_+\nabla_{--}\lambda_+ - 2\,\bar\epsilon_-\nabla_{++}\lambda_- + 2\,\epsilon_+\nabla_{--}\bar\lambda_+ + 2\,\epsilon_-\nabla_{++}\bar\lambda_-
\end{aligned}
\tag{A.1}
$$

with $F_{-+} = [\nabla_{--}, \nabla_{++}] = \frac{1}{2}v_{03}$.

Chiral multiplet:

$$
\begin{aligned}
\delta\Phi_s &= -i\,\epsilon_+\Psi_{-,s} + i\epsilon_-\Psi_{+,s} \\
\delta\Psi_{+,s} &= i\epsilon_+ H_s - 4\,\bar\epsilon_-\nabla_{++}\Phi_s \\
\delta\Psi_{-,s} &= i\epsilon_- H_s + 4\,\bar\epsilon_+\nabla_{--}\Phi_s \\
\delta H_s &= -4\,\bar\epsilon_+\nabla_{--}\Psi_{+,s} - 4\,\bar\epsilon_-\nabla_{++}\Psi_{-,s}\,.
\end{aligned}
\tag{A.2}
$$

Choosing $\bar\epsilon_- = \epsilon_- = 0$, $\bar\epsilon_+ = -\epsilon_+ = i\xi$ one finds:

$$
\begin{aligned}
Q\,v_{++} &= \bar\lambda_+ - \lambda_+ & Q\,(\bar\lambda_+ - \lambda_+) &= -4i\,[\nabla_{--}, \nabla_{++}] \\
Q\,(\bar\lambda_+ + \lambda_+) &= 2\,D & Q\,D &= -2i\nabla_{--}(\lambda_+ + \bar\lambda_+) \\
Q\,B &= \lambda_- & Q\,\lambda_- &= -4i\nabla_{--}B \\
Q\,\bar B &= -\bar\lambda_- & Q\,\bar\lambda_- &= 4i\nabla_{--}\bar B \\
Q\,v_{--} &= 0 & & \\
Q\,\Phi_s &= -\Psi_{-,s} & Q\,\Psi_{-,s} &= 4i\nabla_{--}\Phi_s \\
Q\,\Psi_{+,s} &= H_s & \delta H_s &= -4i\nabla_{--}\Psi_{+,s}\,.
\end{aligned}
\tag{A.3}
$$

Finally, compactifying to zero dimensions and identifying

$$
\begin{aligned}
\phi &= 4iv_{--} = -4i\nabla_{--} & & \\
\bar\phi &= 4iv_{++} = 4i\nabla_{++} & \eta &= 4i(\bar\lambda_+ - \lambda_+) \\
H_{\mathbb{R}} &= 2\,D & \chi_{\mathbb{R}} &= (\bar\lambda_+ - \lambda_+) \\
B &= B_4 & \lambda_- &= \mathcal{M}_4 \\
\Phi_s &= (B_{1,2,3}; \omega_{\dot\alpha}) & \Psi_{-,s} &= -(\mathcal{M}_{1,2,3}; \mu_a) \\
H_s &= (H_{14,24,34}; H_{\dot a}) & \Psi_{+,s} &= (\chi_{14,24,34}; \mu_{\dot a})
\end{aligned}
\tag{A.4}
$$

one finds (2.1).

B. $\mathrm{SU}(2)\times\mathrm{SU}(2)$ with one bifundamental

For the $\mathbb{Z}_p$ orbifold we find in general a $\prod_r \mathrm{U}(N_r)$ gauge theory with bifundamental matter. In particular for the choice $q_\alpha = 0, q_{\hat\alpha} = 1$, $N_r = 0$ for $r > 1$ one finds again a SYM theory with gauge group $\mathrm{U}(N_0)\times\mathrm{U}(N_1)$ but now with a single hypermultiplet. Indeed only one of the two combinations, say $a_{r\hat r} + m$, survives the orbifold projection and formulae (3.7) get modified by the replacements

$$(a_{r\hat r}+x)^2 - m^2 \to (a_{r\hat r}+x) + m\,. \tag{B.1}$$

This results into a single contribution in the numerators of $S_r(x), S_{\hat r}$ in (3.7) corresponding to the single bifundamental in the resulting quiver. This is the case considered in [17]. The result (3.8), (3.7) are indeed in perfect agreement with formula (27,28) in [17] after (B.1) is taken into account (see also [19]). Our formulae can also be tested against the results in [18] for the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ with a single bifundamental matter. Given that the v.e.v.'s in the two gauge groups are $a_1 = -a_2 = a$ and $a_{\hat 1} = -a_{\hat 2} = \hat a$, up to $k = 3$ one finds

$$\begin{aligned}
\mathcal{F} = {} & \frac{(\hat a^2 - a^2)q}{2\,a^2} + \frac{(a^2-\hat a^2)\hat q}{2\,\hat a^2} - \frac{(a^4 - 6\,a^2\,\hat a^2 + 5\,\hat a^4)q^2}{64\,a^6} - \frac{(a^2+\hat a^2)q\,\hat q}{4\,a^2\,\hat a^2} + \\
&+\frac{(a^4-6\,a^2\,\hat a^2+5\,\hat a^4)q^2\,\hat q}{64\,a^6\,\hat a^2} - \frac{(5\,a^4 - 6\,a^2\,\hat a^2 + \hat a^4)\hat q^2}{64\,\hat a^6} + \frac{(5\,a^4-6\,a^2\,\hat a^2+\hat a^4)q\,\hat q^2}{64\,a^2\,\hat a^6} + \\
&+\frac{a^2(9\,a^4 - 14\,a^2\,\hat a^2 + 5\,\hat a^4)\hat q^3}{192\,\hat a^{10}} + \frac{\hat a^2(9\,\hat a^4 - 14\,\hat a^2\,a^2 + 5\,a^4)q^3}{192\,a^{10}} + \\
&+\epsilon_1^2\Bigg[\frac{(\hat a^2-a^2)(a^2-2\,\hat a^2)q^2}{64\,a^8} + \frac{(a^2-\hat a^2)(\hat a^2 - 2\,a^2)\hat q^2}{64\,\hat a^8} + \\
&\quad + \frac{\hat a^2(\hat a^2-a^2)(16\,\hat a^2 - 11\,a^2)\,q^3}{192\,a^{12}} + \frac{a^2(a^2-\hat a^2)(16\,a^2-11\,\hat a^2)\hat q^3}{192\,\hat a^{12}} + \\
&\quad + \frac{(a^4 - 3\,a^2\,\hat a^2 + 2\,\hat a^4)\,q^2\,\hat q}{64\,a^8\,\hat a^2} + \frac{(\hat a^4 - 3\,a^2\,\hat a^2 + 2a^4)\,q\,\hat q^2}{64\,a^2\,\hat a^8}\Bigg] + \cdots\,.
\end{aligned} \tag{B.2}$$

C. $\mathrm{SU}(2)\times\mathrm{SU}(2)$ with two bifundamentals

In this appendix we write the prepotential for the case of a supersymmetric $\mathcal{N} = 2$ SUSY quiver theory with two massless hypermultiplet in the bifundamental with its first gravitational correction using the same notations of the previous paragraph.

$$\begin{aligned}
\mathcal{F} = {} & -\frac{(a^2-\hat a^2)^2\,q}{2\,a^2} - \frac{(a^2-\hat a^2)^2\,\hat q}{2\,\hat a^2} - \frac{\left(a^2-\hat a^2\right)^2(13\,a^4 - 2\,a^2\,\hat a^2 + 5\,\hat a^4)q^2}{64\,a^6} + \\
&+\frac{(a^2-\hat a^2)^2(a^2+\hat a^2)q\,\hat q}{2\,a^2\,\hat a^2} - \frac{(a^2-\hat a^2)^2(5\,a^4 - 2\,a^2\,\hat a^2 + 13\,\hat a^4)\hat q^2}{64\,\hat a^6} - \\
&-\frac{(a^2-\hat a^2)^2\,(23\,a^8 - 6\,a^6\,\hat a^2 + 16\,a^4\,\hat a^4 - 10\,a^2\,\hat a^6 + 9\,\hat a^8)\,q^3}{192\,a^{10}} - \\
&-\frac{(a^2-\hat a^2)^2(9\,a^8 - 10\,a^6\,\hat a^2 + 16\,a^4\,\hat a^4 - 6\,a^2\,\hat a^6 + 23\,\hat a^8)\,\hat q^3}{192\,\hat a^{10}} - \\
&-\frac{(a^2-\hat a^2)^2(a^6 + 9\,a^4\,\hat a^2 + 11\,a^2\,\hat a^4 - 5\,\hat a^6)\,q^2\,\hat q}{32\,a^6\,\hat a^2} +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(a^2-\hat{a}^2)^2(5\,a^6-11\,a^4\,\hat{a}^2-9\,a^2\,\hat{a}^4-\hat{a}^6)q\,\hat{q}^2}{32\,a^2\,\hat{a}^6}+\\
&+{\epsilon_1}^2\left[\frac{\hat{a}^2(a^2-\hat{a}^2)^3\,q^2}{32\,a^8}-\frac{a^2\,(a^2-\hat{a}^2)^3\,\hat{q}^2}{32\,\hat{a}^8}-\frac{(a^4+6\,a^2\,\hat{a}^2+\hat{a}^4)q\,\hat{q}}{4\,a^2\,\hat{a}^2}+\right.\\
&\quad+\frac{(a^2-\hat{a}^2)^3(3\,a^4+7\,a^2\,\hat{a}^2-2\,\hat{a}^4)\,q^2\,\hat{q}}{32\,a^8\,\hat{a}^2}+\frac{(\hat{a}^2-a^2)^3(3\,\hat{a}^4+7\,\hat{a}^2\,a^2-2\,a^4)\,\hat{q}^2\,q}{32\,\hat{a}^8\,a^2}+\\
&\quad+\frac{\hat{a}^2\,(a^2-\hat{a}^2)^3(3\,a^4-3\,a^2\,\hat{a}^2+8\,\hat{a}^4)q^3}{96\,a^{12}}+\\
&\quad\left.+\frac{a^2\,(\hat{a}^2-a^2)^3(3\,\hat{a}^4-3\,\hat{a}^2\,a^2+8\,a^4)\hat{q}^3}{96\,\hat{a}^{12}}\right]+\cdots\,.
\end{aligned}
$$

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